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PARAMETRIC CONTROL OF OSCILLATIONS AND ROTATIONS OF A COMPOUND PENDULUM (A SWING)†

L. D. AKULENKO

Moscow

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Parametric control of plane oscillations and rotations of a rigid body (a plane compound pendulum) is considered. The control is achieved by rectilinear displacements of a point mass with controlled velocity attached to the rigid body. A mathematical model is constructed and control and optimization problems are posed. An approximate asymptotic approach, based on a combination of the averaging method and the maximum principle, is proposed and applied. Rational control rules are constructed and the evolution of the system is analysed. The limiting cases of small oscillations and rapid pendulum rotations are studied.

1. CONSTRUCTION OF THE MATHEMATICAL MODEL AND STATEMENT OF THE CONTROL PROBLEM

CONSIDER the plane motion of a two-mass system consisting of a compound pendulum M coupled to a material point m, see Fig. 1. We assume that the mass m can be displaced in a specified manner relative to the body along the OM axis joining the axis of rotation O to the centre of mass M. Let Φ be the angular inclination of the pendulum, and l the relative coordinate of the point on the OM axis. We then have the following expressions for the kinetic energy K and potential energy Π of the pendulum and the point

$$\mathrm{K}_M = \tfrac{1}{2} J_0 \Phi^{\cdot 2}, \quad \mathrm{K}_m = \tfrac{1}{2} m(l^2\Phi^{\cdot 2} + l^{\cdot 2}) \tag{1.1}$$

$$\Pi_M = Mgl(\Gamma - \cos\Phi), \quad \Pi_m = mgl(\gamma - \cos\Phi)$$

Here $J_0 = J_M + ML^2$ is the moment of inertia of the pendulum about the O axis and J_M the moment of inertia about the centre of mass M, M is the mass, L the moment arm, and g is the acceleration due to gravity. The notation Γ and γ has been introduced in (1.1). $\gamma = 0, \pm 1$. Usually $\Gamma = 1$, i.e. for $\Phi = 0$ the potential energy $\Pi_M = 0$. Sometimes one puts $\Gamma = 0$ or -1, which makes no difference to the equations of motion and is done for convenience. One should put $\gamma = 0$ if l is a phase variable; for a given function $l(t)$ the quantity γ, like Γ, can be chosen arbitrarily.

We will take Φ to be a phase variable and l to be a parameter that changes in a specified manner (see below). As a result we obtain the Lagrange equation for Φ

$$(J_0 + ml^2)\Phi^{\cdot\cdot} + (ML + ml)g\sin\Phi = -2mll^{\cdot}\Phi^{\cdot} \tag{1.2}$$

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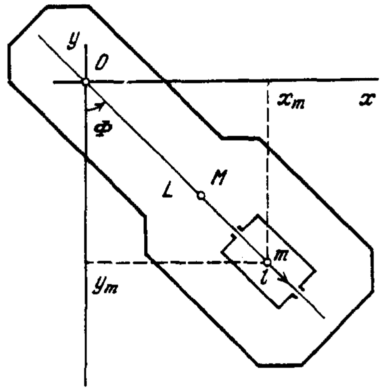

FIG. 1.

For $l=\text{const}$ ($l^{\bullet}\equiv 0$) relation (1.2) turns into the equation of motion of a plane compound pendulum, integrable in terms of elliptic Jacobi functions. The case $l=l(\varepsilon t)$ is of interest in applications, i.e. l is a slowly varying parameter (with rate of change $l^{\bullet}\sim\varepsilon$, where $\varepsilon\ll 1$ is a small parameter). It is usual to take as the unit of time some characteristic quantity associated with the motion of the pendulum, for example the period of small oscillations. The evolution of the oscillations and rotations can be effectively investigated using asymptotic methods of non-linear mechanics (the averaging method [1, 2], etc.).

The rational statement and solution of the problem of controlling the oscillations and rotations of the pendulum by means of displacements of the mass m, i.e. variation of the parameter l in system (1.2), is of considerable interest in its theoretical and applied aspects. A special case of parametric control for a single-mass system ($M=0$) where the point m had its displacement velocity regulated along a massless rod (i.e. a simple pendulum of varying length), was considered in [3]. Here we change to the statement of the control problem for the motions of a system with the more general form (1.2).

We will assume that the rate of change υ of the parameter l is a controlling action from a defined class, and, in particular, that it can take any values within given limits $\upsilon_{1,2}$

$$l^{\bullet}=\upsilon,\quad \upsilon=\upsilon(t),\quad \upsilon_1\leqslant\upsilon(t)\leqslant\upsilon_2 \tag{1.3}$$

Here $\upsilon(t)$ is an integrable function, the constants $\upsilon_{1,2}$ satisfy the conditions $\upsilon_1<0$, $\upsilon_2>0$, and, as is usual, from now on we put $-\upsilon_1=\upsilon_2=\upsilon_0$. The problem of implementing such a kinematic control of the motion of a mass m requires additional considerations. The analysis of [3] has shown that electromechanical control should have negligibly small delay times compared with a typical pendulum oscillation or period of rotation. We also note that under realistic conditions the limits of the variation of l are also bounded: $l_1\leqslant l(t)\leqslant l_2(t)$, in particular $|l(t)-l^0|\leqslant\varepsilon l^0$, i.e. the statement of the control problem should contain phase restrictions.

We note the following fundamental property of the parametrically controlled system (1.2), (1.3). For $\Phi^{\bullet}(t_0)=0$, $\Phi(t_0)=\pi n$, $n=0,\pm1,\ldots$, it is uncontrollable for all $t_0\leqslant t<\infty$. This means that in the lower or upper positions of equilibrium of the pendulum there is an infinitely long time with arbitrary control $\upsilon(t)$ and variation of $l(t)$. The lines $(\pi n, 0, l)$ are invariant in the three-dimensional phase space $(\Phi, \Phi^{\bullet}, l)$. However, one can achieve $\upsilon(t)$ for which these states become Lyapunov-unstable. Then external perturbations, acting on the system in realistic conditions, take it out of the equilibrium position to a sufficient extent and an effective control process ("build-up") begins. If the equilibrium position is stable, and the displacements are small, the build-up requires significant (asymptotically large) time and expenditure of control resources (see below). Hence, in such situations it is desirable to have displacements of the point m perpendicular to the OM axis, and these are often used in practice.

We will therefore formulate the problem of taking the pendulum (1.2) into the required state of oscillation or rotation in a finite time by varying the parameter l satisfying (1.3). The final value of l may also be fixed. Here for control in the form of a program $\upsilon(t)$ or synthesis $\upsilon(t, \Phi, \Phi^{\bullet}, l)$ one can also impose additional optimality conditions in terms of some criterion (time-optimal, least "work", etc.). The construction of the control and the analysis of motion in the general case are very laborious and can be performed numerically, which requires many complex calculations of the boundary-value problem associated with the maximum principle [4]. This is not a constructive approach.

Here we propose to use an efficient approach [3] based on combining the mathematical techniques of the maximum principle [4] and the averaging method [1, 2]. For this we assume that the controlling action υ is in some sense small, and the control process is accomplished over an asymptotically large period of time. In such a control there will be many (in practice several) oscillations or rotations of the pendulum and a significant change in the slowly changing characteristics of the motion [3]. We will formalize the proposed asymptotic approach by introducing a small numerical parameter ε and dimensionless variables and system parameters

$$\theta = \nu t, \quad \nu^2 = \frac{ML + ml_0}{J_0 + ml_0^2} g \equiv \frac{ML}{J_0} \frac{1+\chi}{1+\mu} g > 0$$
$$\sigma = l l_0^{-1}, \quad l_0 > 0; \quad \varepsilon = \upsilon_0 (\nu l_0)^{-1}, \quad \varepsilon u \equiv \upsilon (\nu l_0)^{-1} \tag{1.4}$$

Here θ is the dimensionless time, ν is the frequency of small oscillations at $l = l_0$, $l_0 l$ is a length-scale for the displacement of the mass m from the O axis ($|l_0|>0$) and μ and χ are inertial-geometric dimensionless parameters ($\mu > 0$; to fix our ideas it is assumed that $1+\chi\sigma > 0$). The small parameter ε in (1.4) is associated with the ratio of the longitudinal and transverse velocities of the point m. Other ways of non-dimensionalizing are possible. As a result system (1.2), (1.3) can be represented in the form

$$\frac{1+\mu\sigma^2}{1+\mu}\Phi'' + \frac{1+\chi\sigma}{1+\chi}\sin\Phi = -2\varepsilon\frac{\mu\sigma}{1+\mu}u\Phi'$$
$$\sigma' = \varepsilon u, \quad |u| \leqslant 1; \quad \theta_0 \leqslant \theta \leqslant \theta_*, \quad \theta_* = \Theta\varepsilon^{-1}, \quad \Theta \sim 1 \tag{1.5}$$

Here we have assumed the symmetry of the constraints (1.3) on υ ($-\upsilon_1 = \upsilon_2 = \upsilon_0$) in order to simplify the calculations. We remark that the dimensionless equations for a simple pendulum of variable length [3] can be obtained from (1.5) by passing to the limit $\mu, \chi \to \infty$ (J_0, $M \to 0$).

2. REDUCTION TO STANDARD FORM OF WEAKLY CONTROLLED SYSTEMS WITH ROTATING PHASE

According to (1.5), when $\varepsilon = 0$ the parameter $\sigma = \text{const}$ (the point is frozen), and a compound pendulum with fixed point m performs oscillations or rotations with constant "energy" E given by the values of Φ, Φ' and σ at some instant of time θ

$$E = D\Phi'^2 + B(1-\cos\Phi) = \text{const}, \quad \sigma = \text{const} \tag{2.1}$$
$$D = \tfrac{1}{2}(1+\mu\sigma^2)(1+\mu)^{-1} > 0, \quad B = (1+\chi\sigma)(1+\chi)^{-1} > 0$$

Here E is the dimensionless "energy" of the oscillations or rotations, taken with respect to the quantity $(J_0 + ml_0^2)\nu^2 = (ML + ml_0)g$; in accordance with Sec. 1, see (1.1), the quantities Γ and $\gamma = 1$, which gives $E = 0$ when $\Phi = \Phi' = 0$. The phase integral specifies the connection between

Φ, E and θ with the help of the elliptic Jacobi functions. It is not written out because its explicit expression is not required in the constructions below.

As in [1–3] we differentiate the integrals (2.1) with respect to θ using the perturbed system (1.5). For the slow variables E and σ we obtain the equations

$$E' = \varepsilon u[-d\Phi'^2 + b(1-\cos\Phi)] \equiv \varepsilon u G(\Phi, E, \sigma) \tag{2.2}$$

$$\sigma' = \varepsilon u, \quad |u| \leqslant 1 \quad (d = \mu\sigma(1+\mu)^{-1}, \quad b = \chi(1+\chi)^{-1})$$

The expression for the function G is obtained after the substitution $\Phi'^2 = D^{-1}[E - B(1-\cos\Phi)]$ in accordance with (2.1). In order to close system (2.2) it is necessary to write out the equation of the perturbed phase and the phase integral given above. However, for the approximate solution of the control problem with the slow variables E, σ one can use the procedure of averaging over phase trajectories of the unperturbed system without using the explicit dependence on the phase [2, 3].

We consider the problem of significant changes in the variables E and σ over the asymptotically long time interval $\theta \in [\theta_0, \theta_*]$

$$E(\theta_{0,*}) = E^{0,*}, \quad \sigma(\theta_{0,*}) = \sigma^{0,*}, \quad I[u] \to \min \tag{2.3}$$

The time θ_* at which the process finishes can be either fixed or determined by the solution of the control problem. The optimality criterion $I[u]$ in (2.3) can either be time-optimal ($I[u] = \theta_*$) or an integral quadratic functional describing the expenditure of "energy" during the control; for fixed θ_* one can put $I[u] = \pm E^*$, etc. We note that the values of $\sigma(\theta_*)$ (and also $\sigma(\theta)$, $\theta \in [\theta_0, \theta_*]$) may be significant (as in the J_0, $M = 0$ case [3]), or they may not, as in some other cases (see below).

We will now discuss the equations of controlled motion in the case of small-amplitude oscillations [3], which can have various orders of smallness with respect to ε. Thus, putting $\Phi = \sqrt{(\varepsilon)}\varphi$, $\varphi \sim 1$ into (1.5), we obtain for φ

$$2D\varphi'' + B\varphi = \frac{\varepsilon}{6}B\varphi^3 - 2\varepsilon\frac{\mu\sigma}{1+\mu}u\varphi' + O(\varepsilon^2) \tag{2.4}$$

If $\Phi = \varepsilon^\lambda\varphi$ with $\lambda \geqslant 1$, the terms linear in Eq. (2.4) remain because the non-linear terms are of order ε^2 and higher. In Eq. (2.4) we change to "amplitude-phase" variables according to the formulae

$$\varphi = A\sin\Psi, \quad \varphi' = A\Omega\cos\Psi, \quad \Omega^2 = \tfrac{1}{2}BD^{-1} = \Omega^2(\sigma) > 0 \tag{2.5}$$

As a result we obtain explicit equations for A, Ψ, σ for the first approximation in ε

$$A' = -\varepsilon u f(\sigma)A\cos^2\Psi + \frac{\varepsilon}{6}\frac{A^3}{\Omega}\sin^3\Psi\cos\Psi, \quad A(\theta_0) = A^0 > 0$$

$$\Psi' = \Omega + \varepsilon u f(\sigma)\cos\Psi\sin\Psi - \frac{\varepsilon}{6}\frac{A^2}{\Omega}\sin^4\Psi, \quad \Psi(\theta_0) = \Psi^0 \tag{2.6}$$

$$\sigma' = \varepsilon u, \quad \sigma(\theta_0) = \sigma^0, \quad |u| \leqslant 1, \quad \theta_0 \leqslant \theta \leqslant \theta_* = \Theta\varepsilon^{-1}$$

$$\Omega = \Omega(\sigma), \quad f(\sigma) = \mu\sigma(1+\mu\sigma^2)^{-1} + \tfrac{1}{2}\chi(1+\chi\sigma)^{-1}$$

For brevity we do not show the dependence of f on μ, χ; we note that $f \to 3(2\sigma)^{-1}$ as μ, $\chi \to \infty$ [3]. System (2.6) is essentially non-linear because the frequency Ω depends on one of the unknown slow variables [2, 3]. It follows from (2.6) that $A(\theta) \equiv 0$ if $A^0 = 0$, and, furthermore, one can achieve an unlimited increase or decrease in the amplitude when $A^0 > 0$, $\theta \to \infty$. We note that the cubic addition to Eq. (2.6) for A does not affect the amplitude to first order in ε because its average over Ψ vanishes. It is also important to note that for small oscillations their amplitude can be changed by the same order of smallness ε for $\theta_* = \Theta\varepsilon^{-1}$ since, in general, the effectiveness of the control decreases as $A \to 0$.

3. APPROXIMATE SOLUTION OF THE PROBLEM OF THE OPTIMAL AND QUASI-OPTIMAL CONTROL OF PENDULUM MOTIONS

3.1. *Control of small oscillations*

Avoiding the singularities of transformation (2.5) we assume that $1+\chi\sigma > 0$; the case of small frequencies Ω and passage through zero requires a separate consideration [5, 6]. We will first consider the problem with a fixed, fairly long process end time and an integral quadratic functional with no restriction on the control u [3]

$$A(\theta_*) = A^*, \quad \sigma(\theta_*) = \sigma^*, \quad I[u] = \frac{\varepsilon}{2}\int_{\theta_0}^{\theta_*} u^2 d\theta \tag{3.1}$$

Using the approach employed in [3] we obtain an averaged maximum principle boundary-value problem for a second-order Hamiltonian system. With an error of $O(\varepsilon)$ we construct an optimal control u^* and the required small variables ρ, σ and the p, q conjugate to them

$$u^* = -pf\cos^2\Psi + q, \quad p = \text{const}, \quad \tau = \varepsilon\theta \in [\tau_0, \Theta] \tag{3.2}$$

$$\rho - \rho^0 = \int_{\sigma^0}^{\sigma} F(p, h, f(\xi))d\xi, \quad \tau - \tau_0 = \pm\int_{\sigma^0}^{\sigma} \frac{d\xi}{[2h + (1/8)p^2 f^2(\xi)]^{1/2}}$$

$$q = \tfrac{1}{2}pf \pm [2h + (1/8)p^2 f^2]^{1/2}, \quad F = (\partial h/\partial p)(\partial h/\partial q)^{-1}$$

$$h = \tfrac{1}{2}[(3/8)p^2 f^2 - pfq + q^2] = \text{const}, \quad \rho \equiv \ln(A/A^*)$$

Here $A^*(\rho^* = 0)$, σ^* and Θ are specified and the unknown variables are the parameters h, the averaged Hamiltonian (over slow time τ), and p the variable conjugate to ρ (p = const because ρ is cyclic.) They are governed by two equations for ρ and σ with $\tau = \Theta$. As a result, the implicit relations (3.2) specify the control u^* in the form of a "partial" program with respect to the slow variables (where the phase Ψ should be measured) or a synthesis if one can construct a solution of the boundary-value problem (3.2) for arbitrary initial conditions ρ^0, σ^0 [3]. The expression for cos Ψ in u^* (3.2) can be replaced in accordance with (2.5): $\cos\Psi = \varphi'(A\Omega)^{-1}$, where $A = [\varphi^2 + (\varphi'/\Omega)^2]^{1/2}$.

If $\sigma(\theta_*) = \sigma^*$ is not fixed in (3.1) and is unknown, then $q(\Theta) = 0$ and from the expression for h with $\tau = \Theta$ we obtain the connection between the parameter h and the other unknowns p, σ^*: $h = (3/16)p^2 f^2(\sigma^*)$. Substituting h into Eqs (3.2) for ρ and σ with $\tau = \Theta$, we obtain two transcendental equations for the unknowns p, σ^*. As in the preceding case, these equations can be solved numerically because these expressions reduce to ultra-elliptical integrals (the denominator under the integral sign containing the square root of a sixth-degree polynomial in σ) and more-complicated functions.

We will now consider the time-optimal optimization problem for system (2.6). Using the approach proposed previously [3] we obtain expressions for an approximate optimal control

u^* and averaged Hamiltonian h

$$u^* = \mathrm{sign}(q - pf\cos^2\Psi) \equiv \mathrm{sign}(r - s\cos 2\Psi)$$
$$h = \begin{cases} |r|, & |k|>1, \quad k = r/s \\ (2/\pi)|s|[(1-k^2)^{1/2} + k\arcsin k], & |k|\leqslant 1 \end{cases} \tag{3.3}$$
$$h, p = \mathrm{const}, \quad r = q - s, \quad s = \tfrac{1}{2}pf(\sigma)$$

Note that k is a slowly-varying parameter. The maximum principle boundary-value problem in slow time τ is obtained from the Hamiltonian $h = h(\sigma, p, q)$, see (3.3)

$$\begin{gathered} \rho' = \partial h/\partial p, \quad p = \mathrm{const}, \quad \sigma' = \partial h/\partial q, \quad q' = -\partial h/\partial\sigma \\ \rho(\tau_0) = \rho^0, \quad \rho(\Theta) = 0, \quad \sigma(\tau_0) = \sigma^0 \quad (\sigma(\Theta) = \sigma^* \vee q(\Theta) = 0) \end{gathered} \tag{3.4}$$

As h does not depend on ρ, and $p = \mathrm{const}$, it is the last two equations of (3.4) for σ, q which are to be integrated. One can therefore use phase-plane methods; furthermore, the autonomous system for σ, q is Hamiltonian, i.e. $h = \mathrm{const}$. By using this integral one can integrate the system completely. If the variable σ varies monotonically, it is convenient to use it as an argument and integrate equations for $dq/d\sigma$ and $d\rho/d\sigma$. In the $|k|>1$ regime the integration is elementary and leads to the expressions

$$\begin{gathered} \rho = \rho^0 - \frac{1}{2}\int_{\sigma^0}^{\sigma} f(\xi)d\xi, \quad q = \frac{1}{2}f(\sigma) + c, \quad c = q^* - \frac{1}{2}pf(\sigma^*) \\ u^* = u_\sigma = \mathrm{sign}\, c, \quad \sigma' = \mathrm{sign}\, c = \mathrm{sign}(\sigma^* - \sigma^0), \quad c = \mathrm{const} \end{gathered} \tag{3.5}$$

This case corresponds to displacing the mass m to a specified position σ^* with maximum velocity $u^* = \pm 1$, with $\rho = 0$ for $\sigma = \sigma^*$, i.e. $\tau = \Theta = \tau_0 + |\sigma^* - \sigma^0|$, which corresponds to the special initial condition ρ^0. If however σ^* is not fixed, then $q^* = 0$ in (3.5) and the sign of p is chosen so that the condition $\rho(\sigma^*) = 0$ is satisfied for some σ^*. This control regime is directed towards the variation of σ. In the more general case $|k|\leqslant 1$ the control has an oscillating bang–bang character

$$u^* = u_\rho = \mathrm{sign}(pf)\mathrm{sign}(k - \cos 2\Psi) \tag{3.6}$$

and leads to purposeful simultaneous variation of the oscillation amplitudes, i.e. the variables ρ and σ. For $|k|\ll 1$ it follows from (3.6) that there is almost no drift for the point m: $\sigma(\theta) \simeq \sigma^0$, and by choosing the sign of p one can obtain the required variation of ρ: $\rho' = -(1/\pi)|f(\sigma^0)|\,\mathrm{sign}\rho^0$. In the general situation control of the motion will contain intervals in both regimes: $|k|>1$ and $|k|\leqslant 1$. The construction of an all-purpose algorithm is extremely difficult. However, the analysis clarifies the mechanism of control by oscillations and displacements and gives the following rational control law, consisting of two stages. In the first stage the $|k|>1$ regime is established, corresponding to the fastest displacement of the point m

$$\begin{gathered} u^* = u_\sigma = \mathrm{sign}(\sigma^* - \sigma^0), \quad \Theta_\sigma = \tau_0 + |\sigma^* - \sigma^0|, \quad \tau \in [\tau_0, \Theta_\sigma] \\ \sigma(\tau) = \sigma^0 + \frac{\sigma^* - \sigma^0}{\Theta_\sigma - \tau_0}(\tau - \tau_0), \quad \rho(\tau) = -\frac{u_\sigma}{2}\int_{\tau_0}^{\tau} f(\sigma(\eta))d\eta + \rho^0 \end{gathered} \tag{3.7}$$

From a time $\tau = \Theta_\sigma$ the $k = 0$ regime is used, according to which (see (3.6))

$$u^* = u_\rho = -\mathrm{sign}(pf\cos 2\Psi), \quad \mathrm{sign}p = -\mathrm{sign}\rho(\Theta_\sigma) \tag{3.8}$$

$$\rho(\tau)=\rho_\sigma-\frac{\rho_\sigma}{\Theta-\Theta_\sigma}(\tau-\Theta_\sigma),\quad \Theta=\Theta_\sigma+\frac{\pi|\rho(\Theta_\sigma)|}{|f(\sigma^*)|},\quad \sigma(\tau)\simeq\sigma^*$$

The quantity $\cos 2\Psi$ can be expressed in terms of Φ, Φ' and Ω.

As a result the total time $\Theta=\tau_0$ required to change the slow variables ρ and σ completely with an error of $O(\varepsilon)$ is

$$\Theta-\tau_0=|\sigma^*-\sigma^0|+\pi|\rho(\Theta_\sigma)|\,|f(\sigma^*)|^{-1} \tag{3.9}$$

If the value of $\sigma(\Theta)$ is unspecified, the quantity σ^* can be chosen in (3.7), (3.8) so that the total time Θ in (3.9) is a minimum. In cases when the additional requirement $\sigma(\tau)\simeq\sigma^0\simeq\sigma^*$ is imposed on u, which often occur in practice (swings), the optimal control regime is (3.8), see [3]. Thus for small oscillations one can use the control of u_ρ to achieve an amplitude variation that is exponential in the time τ: $A=A^*\exp\rho(\tau)$, $\rho(\Theta)=0$, where by (3.8) $\rho(\tau)$ is a linear function of $\tau=\varepsilon\theta$ and is negative in the problem of increasing the oscillation amplitude ($A<A^*$) and positive for their damping ($A>A^*$). It follows from the expression for u_ρ that the control changes sign four times at the values $\Psi_i=(2i+1)(\pi/4)$, $i=0, 1, 2, 3$ at times corresponding to the mean phase between the maximum and minimum deflections. At these times the displacement of the mass m is extremal; between neighbouring points its displacement proceeds uniformly, see Fig. 2.

3.2. *Control of large oscillations*

We will use the approach described above to control in succession the slow variables E and σ described by relations (2.2) and (2.3). At the initial stage $\tau\in[\tau_0, \Theta_\sigma]$ we have, like (3.7), $\sigma=\sigma(\tau)$, $u^*=u_\sigma$ and $\Theta_\sigma=\tau_0+|\sigma^*-\sigma^0|$. In this case the energy E varies according to the averaged Eq. (2.2)

$$\begin{aligned}
&E'=u_\sigma G_0(E,\sigma(\tau))\equiv u_\sigma G_0^*(E,\tau),\quad E(\tau_0)=E^0\\
&G_0\equiv\langle G\rangle=b-bD^{-1}(E-B)-(b+dD^{-1}B)\Lambda\\
&\Lambda\equiv\langle\cos\Phi\rangle=\frac{4}{T_0}\left(\frac{D}{B}\right)^{1/2}\int_0^{\Phi^*}\frac{\cos\Phi\,d\Phi}{(\cos\Phi-\cos\Phi^*)^{1/2}}\\
&T_0=T_0(E,\sigma)=4\left(\frac{D}{B}\right)^{1/2}\int_0^{\Phi^*}\frac{d\Phi}{(\cos\Phi-\cos\Phi^*)^{1/2}}\\
&\Phi^*=\Phi^*(E,\sigma)=\arccos(1-EB^{-1}),\quad \sigma=\sigma(\tau)
\end{aligned} \tag{3.10}$$

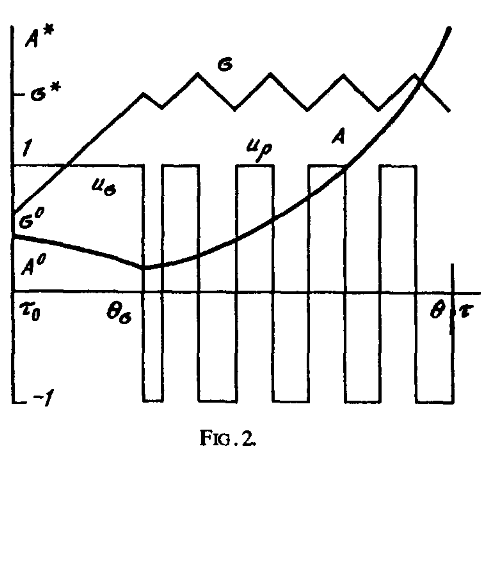

FIG. 2.

Here Φ^* is the maximum amplitude of the oscillations and T_0 is the period; T_0 and Λ are expressed in terms of complete elliptic integrals. The coefficients D, B and d, b are given in (2.1) and (2.2), respectively; they depend on σ and other parameters. The Cauchy problem for (3.10) is integrated for $\tau\in[\tau_0, \Theta_\sigma]$: $E=E(\tau)$, $E_\sigma=E(\Theta_\sigma)$.

In the concluding $\tau\in(\Theta_\sigma, \Theta]$ stage we choose the controlling function $u^*=u_E$ like $u^*=u_p$ in (3.8) so that the point m does not drift, i.e. $\sigma'\simeq\langle u_E\rangle\equiv 0$, but the required oscillation energy change is accomplished $(0<E<2B)$. To this end we represent the function G of (2.2) in the form

$$G(\Phi, E, \sigma)=\Delta(\Phi, E, \sigma)+G_*(E, \sigma), \quad \Delta\equiv G-G_* \tag{3.11}$$

The function Δ has the form $\Delta=a-b\cos\Phi$, where a and b are functions of E and σ, with a, i.e. G_*, being chosen so that the time intervals for positive and negative values of Δ are equal to $T_0/2$ over the period T_0. Using the periodicity and symmetry properties of the function G with respect to Φ, we define the function G_* in the form

$$\begin{gathered} G_*(E, \sigma)=G(\Phi_*, E, \sigma), \quad \Phi_*=\Phi_*(E, \sigma)<\Phi^*(E, \sigma) \\ \int_0^{\Phi_*}\frac{d\Phi}{(\cos\Phi-\cos\Phi^*)^{1/2}}=\frac{T_0}{8}\left(\frac{B}{D}\right)^{1/2}, \quad \cos\Phi^*(E,\sigma)=1-\frac{E}{B} \end{gathered} \tag{3.12}$$

The root Φ_* of Eq. (3.12) ensures the equality of the time intervals for Δ mentioned above. We now impose the control $u^*=u_E$

$$u^*=u_E=\operatorname{sign}(E^*-E_\sigma)\operatorname{sign}\Delta(\Phi, E, \sigma^*) \tag{3.13}$$

We will obtain the solution of the problem for E to a first approximation in ε

$$\begin{gathered} \tau-\Theta_\sigma=\operatorname{sign}(E^*-E_\sigma)\int_{E_\sigma}^{E}\frac{d\xi}{\Delta_0(\xi,\sigma^*)}, \quad \Theta=\tau\big|_{E=E^*} \\ \Delta_0(E, \sigma)=\langle|\Delta(\Phi, E, \sigma)|\rangle \end{gathered} \tag{3.14}$$

where averaging over θ is replaced by averaging over Φ as in (3.10). The total time Θ in (3.14) can be minimized with respect to σ^* if $\sigma(\Theta)$ is unspecified. The control regime (3.13) and (3.14) will be quasi-optimal if one requires in addition that $\sigma(\tau)\simeq\sigma^0\simeq\sigma^*$. Control regimes (3.10), (3.13) and (3.14) contain the implicit assumption that the pendulum only completes one oscillatory motion in $\tau\in[\tau_0, \Theta]$, i.e. $E<2B$. They have to be supplemented by an analysis of rotational motions.

3.3. *Control of the rotations of a pendulum*

For $E>2B$ (see (2.1)) the pendulum will perform non-stopping rotational motions because $\Phi'\neq 0$. It is not automatically the case that $E<2B$ for all $\tau\in[\tau_0, \Theta_0]$ at the first stage of control of large oscillations given by (3.10), i.e. that the pendulum only performs oscillations. Entry into the rotating domain is possible in which the angular variable Φ changes by $\pm 2\pi$ in the period T_0

$$\begin{gathered} T_0(E,\sigma)=\left(\frac{D}{E}\right)^{1/2}\int_0^{2\pi}[1-\beta(1-\cos\Phi)]^{-1/2}d\Phi=2\pi(D/B)^{1/2}\beta^{1/2}[1+\gamma_1\beta+\gamma_2\beta^2+\ldots] \\ \beta=B/E<1/2, \quad \gamma_1=1/2, \quad \gamma_2=9/16,\ldots \end{gathered} \tag{3.15}$$

For "fast rotations" $T_0\simeq 2\pi(D/E)^{1/2}\to 0$ as $\beta\to 0$. The approximate solution of the unperturbed equation (1.5) in powers of $\beta < \frac{1}{2}$ is constructed as in [7]. Quasi-optimal control of a rotating pendulum can be achieved by analogy with Secs 3.1 and 3.2. At the initial stage $\tau \in [\tau_0, \Theta_\sigma]$ for the point m we have $u^* = u_\sigma$, $\sigma = \sigma(\tau)$ (3.7). The rotational energy E varies as given by (3.10) where

$$\Lambda \equiv \langle \cos\Phi \rangle = \frac{1}{T_0} \int_0^{2\pi} \frac{\cos\Phi d\Phi}{\Phi'(\Phi, E, \sigma)} \tag{3.16}$$

The velocity Φ' is given by (2.1); the variable $E = E(\tau)$ is now assumed to be known as a solution of the Cauchy problem (3.10) for $\tau \in [\tau_0, \Theta_\sigma]$. At the final stage $\tau \in (\Theta_\sigma, \Theta]$ we use the $u^* = u_E$ regime (3.13) in which the function $G_* = G(\Phi_*, E, \sigma)$, where $\Phi_* = \Phi_*(E, \sigma)$ is the root of the equation

$$\frac{T_0}{4}\left(\frac{E}{D}\right)^{\frac{1}{2}} = \int_0^{\Phi_*} [1 - \beta(1 - \cos\Phi)]^{-\frac{1}{2}} d\Phi \tag{3.17}$$

For fast rotations we have the asymptotic behaviour $\Phi_* = \pi/2 + \beta/2 + O(\beta^3)$. The equations of motion (2.2) have the form

$$E' = \varepsilon u\left[b - \frac{d}{D}(E - B) - \left(b + \frac{d}{D}B\right)\cos\Phi(\psi, \beta)\right]$$

$$\Phi = \Phi(\psi, \beta) = \psi + \delta\sin\psi + \tfrac{1}{4}\delta^2 \sin 2\psi + O(\delta^3)$$

$$\delta = \delta(\beta) = \tfrac{1}{2}\beta(1 + \gamma_1\beta + \gamma_2\beta^2 + O(\beta^3))^2$$

$$\Phi_* = \Phi(\pi/2, \beta) = \pi/2 + \delta(\beta) + O(\delta^3)$$

where ψ is the phase of the rotational motion. At the first stage $u^* = u_\sigma$, $\tau \in [\tau_0, \Theta_\sigma]$ the energy E is described by the abbreviated averaged equation

$$E' = u_\sigma\left[b - \frac{d}{D}(E - B) + \frac{\beta}{4}\left(b + \frac{d}{D}B\right)\right], \quad E(\tau_0) = E^0 \tag{3.18}$$

with an error of $O(\varepsilon^2 + \beta)$. This can be integrated by perturbation methods with an error of $O(\beta^2)$: $E = E(\tau)$. After a time $\tau = \Theta_\sigma$ the completion stage of controlling the energy from the state $E_\sigma = E(\Theta_\sigma)$ to $E^* = E(\Theta)$ occurs with no drifting of the point m. We have the following expressions ($\sigma \simeq \sigma^*$)

$$u^* = u_E = -\operatorname{sign}(E^* - E_\sigma)\operatorname{sign}(b + dD^{-1}B) \times \operatorname{sign}[\cos\Phi(\psi, \beta) - \cos\Phi(\pi/2, \beta)]$$

$$\cos\Phi(\psi, \beta) - \cos\Phi(\pi/2, \beta) = \cos\psi - \tfrac{1}{2}\beta\cos^2\psi + O(\beta^2) \tag{3.19}$$

$$E(\tau) = E_\sigma + \frac{E^* - E_\sigma}{\Theta - \Theta_\sigma}(\tau - \Theta_\sigma), \quad \Theta = \Theta_\sigma + \frac{\pi}{2}\frac{|E^* - E_\sigma|}{|b + dD^{-1}B|}$$

It follows from (3.19) that in the fast rotation case one can only achieve a power-law (in particular, a linear) variation in the energy and angular velocity of the pendulum with respect to τ.

In the general case the proposed rational approach to controlling the motions of a pendulum can include oscillations and rotations at both stages. This requires the application of regimes described above in Secs 3.1–3.3. As has been established, the efficiency of parametric control is very small at small deviations from the equilibrium position, and this is well known in practice. It would be of interest to study the control of pendulum oscillations and rotations when there are more ways of controlling the displacements of the point m relative to the pendulum, and to construct a rational control regime for such a situation. For example, one could consider two-dimensional motion (longitudinal and transverse) in some domain, oscillatory and rotational (double pendulum) motions, relative rotations by a rotor, etc., including taking into account the dynamics of the driving devices displacing the internal masses. Note that the results obtained carry over to the case when the internal mass m also has significant geometric dimensions, i.e. its central moment of inertia J_m is comparable with J_0. If there are no relative rotations and the centre of mass can only be displaced along the line OM, see Fig. 1, then taking J_m into account amounts to adding $\frac{1}{2}J_m\Phi^{\cdot 2}$ to the expression for K_m (1.1), and setting J_0 equal to $J_0{}^* = J_M + ML^2 + J_m$ throughout what follows.

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